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Spacetime Emergence in the Robertson-Walker Universe from a Matrix model

Johanna Erdmenger*, René Meyer* and Jeong-Hyuck Park†

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Using a novel, string theory-inspired formalism based on a Hamiltonian constraint, we obtain a conformal mechanical system for the spatially flat four-dimensional Robertson-Walker Universe. Depending on parameter choices, this system describes either a relativistic particle in the Robertson-Walker background, or metric fluctuations of the Robertson-Walker geometry. Moreover we derive a tree-level $\mathcal{M}$-theory matrix model in this time-dependent background. Imposing the Hamiltonian constraint forces the spacetime geometry to be fuzzy near the big bang, while the classical Robertson-Walker geometry emerges as the Universe expands. From our approach we also derive the temperature of the Universe interpolating between the radiation and matter dominated eras.

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Recent astronomical data show that the current Universe is very close to the spatially flat Robertson-Walker (RW) geometry [1]. The Universe has evolved from a big bang singularity, near which quantum effects are expected to have played an important role. While a complete quantum gravity description of the big bang is unavailable, effective matrix model descriptions of string/$\mathcal{M}$-theory on time dependent backgrounds have lead to a number of insights [2, 3]. However, the main focus of these studies so far has been on time-dependent and supersymmetry-preserving orbifold or plane wave backgrounds, but not on the physically relevant supersymmetry-breaking RW geometry. One technical obstacle is that the latter lacks a null isometry. Hence the conventional light-cone quantization is not applicable and a new approach is required. We develop such an approach in the present Letter: the characteristic feature of our formalism is the presence of a *Hamiltonian constraint*, *i.e.* a vanishing energy constraint. Instead of fixing the light-cone momentum, we consider a sector of fixed Hamiltonian density. In this way, for the first time it becomes possible, at least at tree level, to construct an M2-brane or $\mathcal{M}$-theory matrix model [4] for the realistic RW geometry and demonstrate the emergence of classical spacetime from an originally fuzzy geometry.

In this Letter we begin with the analysis of the geodesic motion of a *single* D-particle in the RW Universe. In particular, we propose a conformal mechanics model invariant under one-dimensional diffeomorphisms. For two different parameter regimes and gauge choices, this mechanical system describes *either* the geodesic motion of a point particle in the spatially flat RW background, *or* homogeneous metric fluctuations around the background. More precisely, in each case we find a conserved quantity and show that any sector of the fixed value of the quantity is described by the conformal mechanics. This is reminiscent of the AdS/CFT correspondence [5], where matter and gravity dynamics are mapped to each other. Here however both the descriptions of matter and gravity descend from the same CFT action. Finally, we derive a matrix model from the action of the bosonic M2-brane in the RW background, giving a many-particle generalization of the single-particle action, as in flat space [4]. We show that imposing the Hamiltonian constraint in the matrix model ensures the emergence of spacetime. Emergence here means that the Hermitian matrices in the matrix model, whose eigenvalues encode the positions of D-particles, become simultaneously diagonalizable far away from singularities, such that the particle positions can be simultaneously measured, and geometric quantities become classical [3].

I. CONFORMAL MECHANICS

In D-dimensional spacetime, requiring both homogeneity and isotropy of the $D-1$ spatial dimensions, the metric is constrained to the RW geometry [6]

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\big[\mathrm{d}\mathbf{x}^2 + \kappa(\mathbf{x}\cdot\mathrm{d}\mathbf{x})^2/(1-\kappa\mathbf{x}^2)\big]\ , \quad (1)$$

where κ is a constant, and $a(t)$ is the only undetermined scale factor depending on the cosmic time t.

The conformal mechanics ("CFT"), which we will show to be closely related to the spatially flat ($\kappa = 0$) RW Universe, is of the general form

$$\mathcal{S}_{\mathrm{CFT}} = \int \mathrm{d}t\left[\tfrac{1}{2}\eta\dot{\varphi}^2 + \tfrac{1}{2}\eta^{-1}\left(\frac{c_1}{\varphi^2} + c_2\right)\right]. \quad (2)$$

Here c_1, c_2 are constants, $\varphi(t)$ is the only dynamical variable and $\eta(t)$ is the inverse of an einbein. Both φ and η transform under one-dimensional diffeomorphism $t \to s(t)$ as $(\varphi(t), \eta(t)) \to (\varphi(s), \eta(s)/\dot{s})$.

Integrating out the auxiliary variable η reduces the action to $\int \mathrm{d}\varphi\sqrt{c_1\varphi^{-2} + c_2}$. On the other hand, fixing the gauge symmetry with an arbitrary function of time, $\eta \equiv \hat{\eta}(t)$, the mechanical system (2) essentially corresponds to the known conformal mechanics [7]. The gauge fixed action is invariant under the transformation $\delta\varphi = \hat{\eta}(f\dot{\varphi} - \frac{1}{2}\dot{f}\varphi)$, where $f(t)$ is given by a solution

*Max-Planck-Institut für Physik, 80805 München, Germany.
†Dept. of Physics, Sogang University, Seoul 121-742, Korea.

of $\frac{\mathrm{d}}{\mathrm{d}t}[\hat{\eta}\frac{\mathrm{d}}{\mathrm{d}t}(\hat{\eta}\dot{f})] = 0$. This third order differential equation has three solutions which form the symmetry algebra $\mathbf{so}(1,2)$ [14].

The Hamiltonian corresponding to the gauge fixed Lagrangian is, with the canonical momentum $p_\varphi = \hat{\eta}(t)\dot{\varphi}$,

$$\mathcal{H}_{\mathrm{CFT}} = \frac{1}{2\hat{\eta}(t)}\left(p_\varphi^2 - \frac{c_1}{\varphi^2} - c_2\right). \tag{3}$$

The physical states must lie on the surface of vanishing energy $\mathcal{H}_{\mathrm{CFT}} \equiv 0$ in the phase space, as implied by the gauge fixing of diffeomorphism invariance. (Note that throughout the Letter, '$\equiv$' denotes gauge fixings or on-shell relations.)

A. CFT for Single D-particle Dynamics

Our starting point is a novel formalism for a generic mechanical system satisfying the following two conditions: (i) The Hamiltonian is given by the inverse of the Lagrangian

$$\mathcal{H}\mathcal{L} = -m^2, \tag{4}$$

where m is a constant mass parameter. This always holds for a square-root relativistic particle Lagrangian of the form $\mathcal{L} = -m\sqrt{1 - g_{ij}(t,x)\dot{x}^i\dot{x}^j}$, after the gauge fixing to identify the worldline affine parameter with the cosmic time. (ii) There exists a conserved quantity with on-shell value ν, such that for the sector of fixed ν the Lagrangian is completely fixed on-shell as a time- and ν-dependent function,

$$\mathcal{L} \equiv e_\nu(t). \tag{5}$$

Then *the square-root free Lagrangian*

$$\mathcal{L}_\nu := \frac{\mathcal{L}^2}{2e_\nu(t)} - \frac{m^2}{e_\nu(t)} + \frac{e_\nu(t)}{2} \tag{6}$$

together with the Hamiltonian constraint equivalently describes the sector of fixed ν [15]. This can be shown by observing that all the canonical momenta of (6) take the same on-shell values as those of $\mathcal{L}$. Further, the Hamiltonian corresponding to $\mathcal{L}_\nu$ reads from (4) and (6)

$$\mathcal{H}_\nu = \left(\mathcal{L}^2 - e_\nu^2\right)/(2e_\nu). \tag{7}$$

The Hamiltonian constraint $\mathcal{H}_\nu \equiv 0$ then implies (5). Henceforth, as an example (see also [8]), we turn to a relativistic particle in four-dimensional RW background (1) in spherical coordinates

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\Big[\mathrm{d}\rho^2 + r^2(\rho)\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\Big],$$

where $r(\rho) = \sqrt{\mathrm{x}^2}$ is given by $\sin(\sqrt{\kappa}\rho)/\sqrt{\kappa}$, ρ or $\sinh(\sqrt{-\kappa}\rho)/\sqrt{-\kappa}$ depending on κ being positive, zero or negative, respectively. After identifying the worldline affine parameter with cosmic time, the point particle or D-particle [16] Lagrangian in the RW background becomes

$$\mathcal{L} = -m\sqrt{1 - a^2(t)\left(\dot{\rho}^2 + r^2(\rho)\dot{\theta}^2 + r^2(\rho)\sin^2\theta\,\dot{\phi}^2\right)}. \tag{8}$$

The canonical momenta for ρ, θ and ϕ are

$$p_\rho = M(t)\dot{\rho}\,,\; p_\theta = M(t)r^2(\rho)\dot{\theta}\,,\; p_\phi = M(t)r^2(\rho)\sin^2\theta\,\dot{\phi}\,, \tag{9}$$

where we set $M(t) := -m^2a^2(t)/\mathcal{L}$ for compact expression. The corresponding Hamiltonian

$$\mathcal{H} = \sqrt{m^2 + a^{-2}(t)\left[p_\rho^2 + r^{-2}(\rho)\left(p_\theta^2 + p_\phi^2/\sin^2\theta\right)\right]}\,,$$

satisfies (4). In spite of the arbitrariness of $a(t)$, the dynamics is *integrable*, as there exist three mutually Poisson-bracket commuting conserved quantities

$$p_\phi \equiv \text{constant}\,, \tag{10}$$

$$J^2 := p_\theta^2 + p_\phi^2/\sin^2\theta \equiv j(j+1)\,, \tag{11}$$

$$p_\rho^2 + J^2/r^2(\rho) \equiv (\nu m)^2\,. \tag{12}$$

The dimensionless constant j plays the role of a classical $\mathbf{so}(3)$ angular momentum. Introducing a time dependent mass as an on-shell value of $M(t)$,

$$m_\nu(t) := ma(t)\sqrt{a^2(t) + \nu^2} \equiv M(t)\,, \tag{13}$$

the Hamiltonian and the Lagrangian assume the on-shell values, $\mathcal{H} = -m^2/\mathcal{L} \equiv m_\nu(t)/a^2(t)$. As for (6) we have

$$\mathcal{L}_\nu = \frac{m_\nu(t)}{2}\left(\dot{\rho}^2 + r^2(\rho)\dot{\theta}^2 + r^2(\rho)\sin^2\theta\,\dot{\phi}^2\right) + \frac{(\nu m)^2}{2m_\nu(t)}\,. \tag{14}$$

Again, all the off-shell canonical momenta of (14) match with the on-shell ones of (8). Further, the corresponding Hamiltonian

$$\mathcal{H}_\nu = \frac{1}{2m_\nu(t)}\left(p_\rho^2 + \frac{J^2}{r^2(\rho)} - (\nu m)^2\right) \tag{15}$$

exhibits the same mutually commuting conserved quantities as (10)-(12). Thus the surface of the vanishing energy $\mathcal{H}_\nu \equiv 0$ in the phase space of the dynamical system (15) describes precisely the relativistic particle in the RW background for a sector of fixed ν. Further the subsector of fixed angular momentum is reached by setting $J^2 \equiv j(j+1)$, which reduces (15) to the conformal mechanics (3). In this way, *the conformal mechanics* (3) *with the choice* $\varphi = \rho$, $\hat{\eta} = m_\nu(t)$, $c_1 = -j(j+1) \leq 0$, $c_2 = (\nu m)^2 \geq 0$ *describes the geodesic motion of a relativistic particle with respect to cosmic time in the spatially flat RW Universe, with fixed conserved quantities* ν, j.

B. CFT for Homogeneous Gravity

Pioline and Waldron [9] observed that for a solely time-dependent, *generic* D-dimensional metric in longitudinal

gauge

$$\mathrm{d}s^2 = -e^2\varrho^{-2}\mathrm{d}t^2 + \varrho^{2/(D-1)}\hat{g}_{ij}\mathrm{d}x^i\mathrm{d}x^j\,,\ \det\hat{g}=1\,, \quad (16)$$

the Einstein-Hilbert action with cosmological constant reduces to a mechanical system for a relativistic "fictitious" point particle,

$$-\int \mathrm{d}^D x\sqrt{-g}\,(R-2\Lambda) = V\int \mathrm{d}t\,\frac{1}{e}\left(\frac{D-2}{D-1}\right)\dot{\varrho}^2 - 2e\left(\frac{\hat{C}}{\varrho^2}-\Lambda\right).$$

Here V is the $D-1$ dimensional spatial volume, henceforth normalized to one, and $\hat{C} := \frac{1}{8}e^{-2}\varrho^4\mathrm{tr}(\hat{g}^{-1}\dot{\hat{g}}\hat{g}^{-1}\dot{\hat{g}})$ which contains a non-linear σ-model metric for the coset $\mathrm{SL}(D-1)/\mathrm{SO}(D-1)$ as $\mathrm{tr}(\hat{g}^{-1}\dot{\hat{g}}\hat{g}^{-1}\dot{\hat{g}}) = h_{ab}(\theta)\dot{\theta}^a\dot{\theta}^b$. In terms of the momenta $p_\varrho = \omega\dot{\varrho}/(4e)$, $p_a = -\frac{1}{2}\varrho^2 e^{-1}h_{ab}\dot{\theta}^b$, we identify $\hat{C} = \frac{1}{2}h^{ab}(\theta)p_a p_b$ as the kinetic energy on the coset space and write the Hamiltonian

$$\mathcal{H}_{\mathrm{E.H.}} = \frac{2e}{\omega}\left(p_\varrho^2 - \frac{\omega\hat{C}}{\varrho^2} - \omega\Lambda\right),\ \ \omega := 8\left(\frac{D-2}{D-1}\right). \quad (17)$$

$\hat{C}$ is conserved [9] and, in fact, positive semi-definite [17].

The original motivation of [9] to consider the homogeneous modes only was based on the observation in [10] that near cosmological singularities, inhomogeneous modes generically decouple. Here we make the further observation that the spatially flat RW geometry (1) is a special case of (16) as $e=\varrho=a^{D-1}$, $\hat{g}_{ij}=\delta_{ij}$, and hence (16) is the most general homogeneous and non-perturbative fluctuation of the RW metric. In the cosmic time gauge $e=\varrho$, *the conformal mechanics (3) with the choice* $\varphi=\varrho$, $\hat{\eta}=\frac{1}{4}\omega a^{1-D}$ *or* $\eta=\omega/(4\varphi)$, $c_1\equiv\omega\hat{C}\geq 0$, $c_2=\omega\Lambda$ *describes the homogeneous metric fluctuations of the spatially flat RW Universe with respect to cosmic time, with fixed value of kinetic energy on the coset space.* In particular, near the big bang ($a\simeq 0$), the choice of small $\hat{\eta} = ma\sqrt{a^2+\nu^2}$ and negative c_1 describes the matter, whilst large $\hat{\eta}=\frac{1}{4}\omega a^{1-D}$ and positive c_1 describes the gravity. As an example for the map between gravity and matter, the metric element $\varrho = \sqrt{\det g_{ij}}$ in gravity is mapped to the radial coordinate ρ of the particle trajectory.

II. MATRIX MODEL FOR RW UNIVERSE

Now we turn to the description of *many* D-particles in the spatially flat RW Universe. The description is generically given by a Yang-Mills quantum mechanics [11], *i.e.* by a matrix model generalization of a single particle action. In a flat background, the coupling of the Yang-Mills potential $[X^i,X^j]^2$ in the matrix model can be freely scaled and therefore is irrelevant. However, in the RW background the coupling coefficient is time-dependent, and cannot be simply deduced from the one-particle action (14). We determine this time-dependent coupling by deriving the tree-level $\mathcal{M}$-theory matrix model from the bosonic M2-brane action in the RW background. The dynamics of an M2-brane with tension T embedded in a D-dimensional target spacetime is governed by the Nambu-Goto action

$$S_{\mathrm{M2}} = -T\int \mathrm{d}^3\xi\sqrt{-\det\Big(\partial_{\hat{\alpha}}x^\mu\partial_{\hat{\beta}}x^\nu g_{\mu\nu}(x)\Big)}\,. \quad (18)$$

The three-dimensional worldvolume of the M2-brane is parameterized by coordinates $\xi^{\hat{\alpha}}$, $\hat{\alpha}=0,1,2$, and the embedding functions are given by $x^\mu(\xi)$, $\mu=0,1,\cdots,D-1$. Adopting the cosmic gauge $t=x^0=\xi^0$ as well as the longitudinal gauge $\partial_t x^\mu\partial_\alpha x^\nu g_{\mu\nu}=0$, $\alpha=1,2$ (see [12] for a detailed procedure), the remaining worldvolume diffeomorphism is, at this stage, static, *i.e.* ξ^0-independent. The Nambu-Goto Lagrangian in the spatially flat RW background then reduces to

$$\mathcal{L}_{\mathrm{M2}} = -Ta^2(t)\sqrt{(1-a^2(t)\dot{x}^2)\det\mathcal{G}}\,, \quad (19)$$

where the determinant $\det\mathcal{G} = \det(\partial_\alpha x^i\partial_\beta x_i)$, $\alpha,\beta=1,2$, is taken over spatial M2-brane coordinates only. Spatial indices i,j are contracted with the flat metric δ_{ij} such that $\dot{x}^2 := \dot{x}^i\dot{x}^j\delta_{ij}$. With the momenta $p_i = Ta^4\dot{x}_i\sqrt{\det\mathcal{G}/(1-a^2\dot{x}^2)}$, the equation of motion reads

$$\dot{p}_i \equiv \partial_\alpha\left[Ta^2\sqrt{(1-a^2\dot{x}^2)\det\mathcal{G}}\,\mathcal{G}^{-1\alpha\beta}\partial_\beta x_i\right]\,, \quad (20)$$

and the longitudinal gauge condition becomes

$$p_i\partial_\alpha x^i = 0 \quad\Longleftrightarrow\quad \dot{x}_i\partial_\alpha x^i = 0\,. \quad (21)$$

In terms of the Nambu bracket $\{x,y\}_{\mathrm{NB}} := \epsilon^{\alpha\beta}\partial_\alpha x\partial_\beta y$ ($\epsilon^{12}=1$, $\epsilon^{\alpha\beta}=-\epsilon^{\beta\alpha}$),the determinant can be expressed as $\det\mathcal{G} = \frac{1}{2}\left\{x^i,x^j\right\}_{\mathrm{NB}}\{x_i,x_j\}_{\mathrm{NB}}$. Further (20) and (21) imply respectively

$$\partial_t(p^2) + T^2a^6\,\partial_t\det\mathcal{G} \equiv 0\,, \qquad \left\{p_i\,,\,x^i\right\}_{\mathrm{NB}} = 0\,. \quad (22)$$

In the following we consider the sector of solution-space with fixed on-shell value of the Hamiltonian density

$$\mathcal{H}_{\mathrm{M2}} = a^{-1}\sqrt{p^2+T^2a^6\det\mathcal{G}} \equiv \Omega(\xi)\,. \quad (23)$$

Since Ω transforms as a scalar density, fixing Ω finally breaks the remaining static diffeomorphisms down to the static area-preserving ones. This gauge-fixed sector is then equally described by a square-root free Lagrangian

$$\mathcal{L}_\Omega := \tfrac{1}{2}\left(\Omega a^2\,\dot{x}^i\dot{x}_i - \Omega^{-1}T^2a^4\det\mathcal{G} + \Omega\right)\,, \quad (24)$$

since the canonical momenta as well as the equations of motion are identical to the on-shell ones of (19). Similarly to the one-particle case in section I, the Hamiltonian constraint for $\mathcal{L}_\Omega$ matches with (23) as

$$\mathcal{H}_\Omega = \left(p^2+T^2a^6\det\mathcal{G} - a^2\Omega^2\right)/\left(2a^2\Omega\right) \equiv 0\,. \quad (25)$$

Identifying $\det\mathcal{G} = m^2/(T^2a^4)$ and $\Omega = m_\nu(t)/a^2(t)$, the M2-brane Lagrangians (19) and (24) would respectively reduce to the point particle ones (8) and (14) for

$\kappa = 0$. However, (22) then would imply that $p^2 - 2m^2a^2$ is conserved and the Hamiltonian constraint (25) could not be met. Therefore, unlike in the flat background [4], in an expanding Universe the M2-brane dynamics cannot be consistently truncated to a point particle dynamics.

The matrix regularization of (24) prescribes to replace the dynamical fields $x^i(t,\xi^\alpha)$ by time-dependent $N \times N$ Hermitian matrices $X^i(t)$, the Nambu bracket $\{x^i, x^j\}_{\rm NB}$ by a matrix commutator $i[X^i, X^j]$ [13], and the worldvolume coordinates ξ^α by non-dynamical matrices $\hat{\xi}^\alpha$ satisfying the non-commutative relation $[\hat{\xi}^\alpha, \hat{\xi}^\beta] = i\epsilon^{\alpha\beta} \times$ constant [12]. The resulting M2-brane matrix model is of the general form

$$\hat{\mathcal{L}}_{\rm M2} = {\rm Tr}\left[\frac{\hat{\Omega}a^2}{2}\left(D_tX^i\right)^2 + \frac{a^4}{l^6\hat{\Omega}}\left[X^i, X^j\right]^2 + \frac{\hat{\Omega}}{2}\right], \quad (26)$$

where $\hat{\Omega}(t) := \Omega(t, \hat{\xi}^\alpha)$ and l is a constant length. The covariant time derivative $D_tX^i = \dot{X}^i - i[A_0, X^i]$ involves a non-dynamical gauge field A_0, such that the matrix model admits a U(N) gauge symmetry. With the canonical momenta $P^i = \frac{1}{2}a^2(\hat{\Omega}D_tX^i + D_tX^i\hat{\Omega})$, δA_0 gives the Gauss constraint for the physical states $[P^i, X_i] = 0$, which is consistent with (22). Combining the result of section I and (26) with the identifications $\hat{\Omega}(t) \equiv m_\nu(t)/a^2(t)$, $\hat{\eta} \equiv m_\nu(t)$ (13), we finally determine the precise form of the matrix model for N D-particles in the flat RW background, for which the Hamiltonian is

$$\hat{\mathcal{H}}_{\rm D0} = \frac{1}{2\hat{\eta}(t)}{\rm Tr}\left[P_i^2 - 2a^6(t)l^{-6}\left[X^i, X^j\right]^2 - (\nu m)^2\right]. \quad (27)$$

One crucial difference between the two matrix models for the M2-brane (26) and for D-particles (27) is the last non-dynamical potential term which gives rise to a different Hamiltonian constraint, reflecting the different dynamical behavior of M2-brane and D-particles. We interpret the non-dynamical term as the temperature $\mathcal{T}$ of the Universe [6], since from (27) and $\hat{\mathcal{H}}_{\rm D0} \equiv 0$ this term corresponds to the average particle energy. For the D-particle case, we have $\mathcal{T} = m\nu^2/(2a\sqrt{a^2+\nu^2})$ which nicely interpolates between the temperatures of the early radiation dominated era, $\mathcal{T} \propto a^{-1}$ and the late matter dominated era, $\mathcal{T} \propto a^{-2}$.

Both in the Hamiltonian corresponding to (26) and in (27), the constraint $\hat{\mathcal{H}} \equiv 0$ gives rise to an explicit realization of spacetime emergence: $-{\rm Tr}[X^i, X^j]^2 = {\rm Tr}[X^i, X^j][X_i, X_j]^\dagger \geq 0$ must decrease on-shell as the scale factor $a(t)$ increases [18]. Near the singularity the X^i do not commute and thus cannot be simultaneously diagonalized. This leaves the particle positions fuzzy, and hence the spacetime geometry they probe. As the scale factor grows or the Universe cools down, the fuzziness disappears and the classical Robertson-Walker geometry emerges. In [3], a similar scenario was found in plane wave backgrounds from the time-dependent coupling of the Yang-Mills potential. Our matrix model is, however, subject to the additional Hamiltonian constraint.

As a generalization of the one-particle case in section I, the inhomogeneous fluctuations in the RW Universe are expected to be mapped to the matrix model (26). Quantum corrections to our scenario can be calculated in analogy to [3], in particular they may restrict the time dependence of $a(t)$.

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[14] With $\beta(t) := \int {\rm d}t\,(l\hat{\eta}(t))^{-1}$, $p_\varphi = \hat{\eta}\dot{\varphi}$, three solutions are $f_0 = \frac{1}{\sqrt{2}}\left(l^2 + \frac{1}{2}\beta^2\right)$, $f_1 = l\beta$, $f_2 = \frac{1}{\sqrt{2}}\left(l^2 - \frac{1}{2}\beta^2\right)$, and the corresponding Noether charges form $\mathbf{so}(1,2)$ Lie algebra, $\{Q_1, Q_2\}_{\rm PB} = -Q_0$, $\{Q_2, Q_0\}_{\rm PB} = Q_1$, $\{Q_0, Q_1\}_{\rm PB} = Q_2$, where $Q_f = \frac{1}{2}f\left(p_\varphi^2 - c_1\varphi^{-2}\right) - \frac{1}{2}\hat{\eta}\dot{f}\varphi p_\varphi + \frac{1}{4}\hat{\eta}\varphi^2\frac{\rm d}{{\rm d}t}(\hat{\eta}\dot{f})$ satisfying $\{\varphi, Q_f\}_{\rm PB} = \delta_f\varphi$.
[15] *cf.* (6) and Polyakov Lagrangian $\mathcal{L}_{\rm Poly} = \frac{1}{2}\left(e^{-1}\mathcal{L}^2 + e\right)$.
[16] Any possible homogeneous dilaton factor can be absorbed into the RW metric *via* redefinition of 'frame'.
[17] With the diagonalization $\hat{g} = o\lambda o^t$, $oo^t = 1$, $\lambda > 0$, ${\rm tr}\left(\hat{g}^{-1}\dot{\hat{g}}\hat{g}^{-1}\dot{\hat{g}}\right) = 2\sum_{i>j}\left[(\sqrt{\lambda_i/\lambda_j} - \sqrt{\lambda_j/\lambda_i})(o^t\dot{o})_{ij}\right]^2 + \sum_i(\dot{\lambda}_i/\lambda_i)^2 \geq 0$.
[18] For (26) note from (22) $\partial_t(a^4/\Omega^2) \equiv 6a\dot{a}p^2/\Omega^4 > 0$.